JOURNAL OF ALGEBRA 133, 424–432 (1990)

Characterizing a Free Group in Its Automorphism Group

EDWARD FORMANEK*

The Pennsylvania State University, University Park, Pennsylvania 16802 and Mathematical Sciences Research Institute Berkeley, California 94720

Communicated by Susan Montgomery

Received December 21, 1988

DEDICATED, WITH GRATITUDE, TO THE MEMORY OF ISRAEL NATHAN HERSTEIN

INTRODUCTION

Let $A(G)$ denote the automorphism group of a group G. If $g \in G$, g induces an inner automorphism $i(g) \in A(G)$, where

$$i(g)(h) = ghg^{-1}$$

for all $h \in G$. The map $i: G \to A(G)$ is a group homomorphism of G onto a normal subgroup $i(G)$ of G. A small calculation shows that

$$\phi i(g) \phi^{-1} = i(\phi(g)),$$

for $\phi \in A(G)$ and $g \in G$.

If the center of G is trivial, then $i: G \to A(G)$ is one-to-one, and G can be identified with a normal subgroup of $A(G)$, with the action on G given by conjugation. A group G is said to be *complete* if $i: G \to A(G)$ is an isomorphism. A theorem of Burnside [Bu, p. 95] says that if G is a group with trivial center, then $A(G)$ is complete if and only if G is a characteristic subgroup of $A(G)$.

Let F be a free group of finite rank $n \geqslant 2$. In [D–F] Joan Dyer and I used Burnside's criterion to show that $A(F)$ is a complete group. More precisely, we showed that if $\phi: A(F) \to A(F)$ is an automorphism of $A(F)$, then $\phi(F) = F$.

* Partially supported by National Science Foundation Grant DMS-88-01967.

0021-8693/90 $3.00

As noted above, F (when identified with the group of inner automorphisms) is a normal subgroup of $A(F)$. The main result of this paper, Theorem 9, says that F is the unique normal subgroup of $A(F)$ which is a free group of rank n. It implies that F is characteristic in $A(F)$, and therefore has the completeness of $A(F)$ as a corollary.

The proof of Theorem 9 follows the same general plan as [D–F], using the kernel $K(F)$ of the canonical homomorphism $A(F) \to A(F/F')$. We assume that G is a normal subgroup of $A(F)$ which is free of rank n, and show in order that $G \subseteq K(F)$, $GK(F)' = FK(F)'$, $G' \subseteq F'$, and finally that $G = F$.

The K-Series of a Free Group

Notation. Let G be a group. If $a, b \in G$, $[a, b] = aba^{-1}b^{-1}$. If A and B are subsets of G, $[A, B]$ denotes the subgroup of G generated by all commutators $\{[a, b] \mid a \in A, b \in B\}$. The ith term of the lower central series of G is denoted $\gamma_i(G)$. ($\gamma_1(G) = G$, $\gamma_2(G) = [G, G] = G'$, $\gamma_{i+1}(G) = [\gamma_i(G), G]$.) $A(G)$ denotes the automorphism group of G.

If C is a characteristic subgroup of G, then there is a canonical homomorphism $A(G) \to A(G/C)$.

Definition. The kernel of the canonical homomorphism $A(G) \to A(G/G')$ is called the group of IA-automorphisms of G and is denoted $K(G)$ (or $K_1(G)$). The kernel of the canonical homomorphism $A(G) \to A(G/\gamma_{i+1}(G))$ is denoted $K_i(G)$.

Let $F = F\langle x_1, ..., x_n\rangle$ be a free group of finite rank $n \geqslant 2$ with a fixed generating set $x_1, ..., x_n$. We will identify F with the inner automorphisms in $A(F)$. *If $\phi \in A(F)$ and $x \in F$, we will employ two different notations for the image of x under ϕ: When we simply regard ϕ as a function ϕ: $F \to F$, then we write $\phi(x)$; when we work inside $A(F)$ and identify F with the inner automorphisms in $A(F)$, then we write $\phi x \phi^{-1}$, which signifies multiplication of group elements inside $A(F)$.*

The following theorem collects results about $A(F)$ which are used later.

Theorem 1. (a) (Nielsen; see [M-K-S, pp. 162–169]) *The canonical map $A(F) \to A(F/F')$ is onto.*

(b) [M-K-S, Exercise 5, p. 347]. $K_i(F) \cap F = \gamma_i(F)$.

(c) [A, Theorem 1.1]. $[K_i(F), K_j(F)] \subseteq K_{i+j}(F)$. *Thus* $\gamma_i(K(F)) \subseteq K_i(F)$.

(d) [A, p. 246]. *$K_i(F)/K_{i+1}(F)$ is torsion-free abelian. Hence $K(F)$ is torsion-free.*

(e) [B, *Lemma* 5]. $K_2(F)=K(F)'$.

(f) [D-F, *Theorem* C]. $\bigcap FK_i(F)=F$.

(g) [D-F, *Theorem* D]. $F=\{\phi\in K(F) \mid [\phi, K(F)]\subseteq F'\}$.

The rest of this section is devoted to proving Theorem 2, which characterizes $FK_2(K)/K_2(F)$ as a subgroup of $K(F)/K_2(F)$.

Theorem 1(a) says that

$$1\to K(F)\to A(F)\to A(F/F')\to 1$$

is exact, or that

$$A(F)/K(F)\cong A(F/F')\cong A(\mathbb{Z}^n)\cong \mathrm{GL}(n, \mathbb{Z}).$$

If $\phi\in K_t(F)$ and $x\in F$, then $[\phi, x]\in\gamma_{t+1}(F)$. The map $x\mapsto [\phi, x]\,\gamma_{t+2}(F)$ is a group homomorphism from F to $\gamma_{t+1}(F)/\gamma_{t+2}(F)$ whose kernel contains F'. Hence there is an induced homomorphism

$$\varepsilon_t(\phi)\colon F/F'\to\gamma_{t+1}(F)/\gamma_{t+2}(F).$$

The map

$$K_t(F)\xrightarrow{\varepsilon_t}\mathrm{Hom}_{\mathbb{Z}}(F/F', \gamma_{t+1}(F)/\gamma_{t+2}(F))$$

defined by $\phi\mapsto\varepsilon_t(\phi)$ is a group homomorphism which respects the action of $A(F)$ by conjugation. The definition of $K_t(F)$ implies that $K_{t+1}(F)$ is the kernel of ε_t. Thus there is an induced one-to-one map, which we also denote ε_t,

$$K_t(F)/K_{t+1}(F)\xrightarrow{\varepsilon_t}\mathrm{Hom}_{\mathbb{Z}}(F/F', \gamma_{t+1}(F)/\gamma_{t+2}(F)). \tag{1}$$

The action of $A(F)$ on both sides of (1) is trivial when restricted to $K(F)$, so ε_t is an $A(F)/K(F)$-homomorphism (i.e., a $GL(n, \mathbb{Z})$-homomorphism).

Let k_{ij} $(1\leqslant i, j\leqslant n,\ i\neq j)$ and k_{ijl} $(1\leqslant i, j, l\leqslant n,\ i\neq j<l\neq i)$ denote the elements of $K(F)$ defined by

$$\begin{aligned} k_{ij}(x_i)&=x_jx_ix_j^{-1}, & k_{ij}(x_s)&=x_s & (s&\neq i);\\ k_{ijl}(x_i)&=x_i[x_j, x_l], & k_{ij}(x_s)&=x_s & (s&\neq i). \end{aligned}\tag{2}$$

The images of the k_{ij}, k_{ijl} under ε_1 map to the obvious basis of $\mathrm{Hom}_{\mathbb{Z}}(F/F', \gamma_2(F)/\gamma_3(F))$ (i.e., the basis which arises by taking $\{x_iF' \mid 1\leqslant i\leqslant n\}$ as basis for F/F' and $\{[x_i, x_j]\,\gamma_3(F) \mid 1\leqslant i<j\leqslant n\}$ as basis for $\gamma_2(F)/\gamma_3(F)$. Thus ε_1 in (1) above (specializing $t=1$) is an isomorphism, a fact proved by Bachmuth [B, Theorem p. 7].

The map $[x, y]\,\gamma_3(F)\mapsto xF'\wedge yF'$ defines an isomorphism between

$\gamma_2(F)/\gamma_3(F)$ and $\Lambda_2(F/F')$, the second exterior power of F/F'. Thus there are isomorphisms,

$$\begin{aligned}\operatorname{Hom}_{\mathbb{Z}}(F/F', \gamma_2(F)/\gamma_3(F)) &\cong \operatorname{Hom}_{\mathbb{Z}}(F/F', \Lambda_2(F/F')) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))\end{aligned}$$

which are compatible with the action of $A(F)/K(F) \cong \mathrm{GL}(n, \mathbb{Z})$. The action of $\mathrm{GL}(n, \mathbb{Z})$ on $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$ is induced from the action of $A(F)/K(F)$ on $\operatorname{Hom}_{\mathbb{Z}}(F/F', \gamma_2(F)/\gamma_3(F))$ from the isomorphism $F/F' \cong \mathbb{Z}^n$ arising from the fixed basis $x_1, \ldots, x_n$ for F. It is the standard action of $\mathrm{GL}(n, \mathbb{Z})$ on $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$, and it is therefore the restriction of the standard action of $\mathrm{GL}(n, \mathbb{Q})$ on $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \Lambda_2(\mathbb{Q}^n))$ to its $\mathrm{GL}(n, \mathbb{Z})$-submodule $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$.

Consider

$$\begin{aligned}K(F)/K_2(F) \xrightarrow{\varepsilon_1} &\operatorname{Hom}_{\mathbb{Z}}(F/F', \gamma_2(F)/\gamma_3(F)) \\ &\cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n)) \subseteq \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \Lambda_2(\mathbb{Q}^n)).\end{aligned} \tag{3}$$

If U is an $\mathrm{SL}(n, \mathbb{Z})$-invariant submodule of $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$, then the fact that $\mathrm{SL}(n, \mathbb{Q})$ is generated by elementary matrices $I + ae_{ij}$ $(a \in \mathbb{Q},\ i \neq j)$ implies that $\mathbb{Q}U$ is $\mathrm{SL}(n, \mathbb{Q})$-invariant. Conversely, if V is an $\mathrm{SL}(n, \mathbb{Q})$-invariant subspace of $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \Lambda_2(\mathbb{Q}^n))$ of dimension m over $\mathbb{Q}$, then $V \cap \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$ is an $\mathrm{SL}(n, \mathbb{Z})$-invariant rank m direct summand of $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$. Thus

$$U \mapsto \mathbb{Q}U, \qquad V \mapsto V \cap \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))$$

define inverse bijections

$$\begin{aligned}&\{\mathrm{SL}(n, \mathbb{Z})\text{-invariant rank } m\ \mathbb{Z}\text{-module direct summands of} \\ &\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}^n, \Lambda_2(\mathbb{Z}^n))\} \leftrightarrow \{\mathrm{SL}(n, \mathbb{Q})\text{-invariant dimension } m \\ &\mathbb{Q}\text{-vector subspaces of } \operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \Lambda_2(\mathbb{Q}^n))\}.\end{aligned} \tag{4}$$

The representation theory of $\mathrm{SL}(n, \mathbb{Q})$ (see [G]; the details are too long to include here) shows that

$$\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \Lambda_2(\mathbb{Q}^n)) \cong \mathbb{Q}^n \oplus V_n$$

as an $\mathrm{SL}(n, \mathbb{Q})$-module, where $\mathbb{Q}^n$ is the standard $\mathrm{SL}(n, \mathbb{Q})$-module, and V_n is an irreducible $\mathrm{SL}(n, \mathbb{Q})$-module of dimension $\frac{1}{2}n(n+1)(n-2)$ (for $n \geqslant 3$; for $n = 2$, $V_2 = 0$). In either case, $\mathbb{Q}^n$ is the unique $\mathrm{SL}(n, \mathbb{Q})$-invariant submodule of $\operatorname{Hom}_{\mathbb{Q}}(\mathbb{Q}^n, \Lambda_2(\mathbb{Q}^n))$ of dimension n over $\mathbb{Q}$. Invoking the bijection of (4) and the isomorphism ε_1 of (3) shows that $K(F)/K_2(F)$ has at most one $A(F)/K(F)$-invariant direct summand of rank n.

On the other hand, in terms of the automorphisms k_{ij} in (2) above,

$$x_i = \prod \{k_{ji} \mid 1 \leqslant j \leqslant n, j \neq i\}.$$

(The k_{ji} in the product actually commute.) Since the $k_{ji}K_2(F)$ are part of a basis for $K(F)/K_2(F)$,

$$FK_2(F)/K_2(F) = gp\langle x_1 K_2(F), ..., x_n K_2(F)\rangle$$

is an $A(F)/K(F)$-invariant rank n $\mathbb{Z}$-module direct summand of $K(F)/K_2(F)$. Thus we have shown

THEOREM 2. *Let $F = F\langle x_1, ..., x_n\rangle$ be free of finite rank n, where $n \geqslant 2$, and let $A(F)$ act by conjugation on $K(F)/K_2(F)$. Then $FK_2(F)/K_2(F)$ is the unique $A(F)$-invariant rank n $\mathbb{Z}$-module direct summand of $K(F)/K_2(F)$.*

PROOF OF THE MAIN THEOREM

For the rest of the paper, we fix the following data:

> *$F = F\langle x_1, ..., x_n\rangle$ is free of* rank $n \geqslant 2$. *G is a free group of* rank *n which is a normal subgroup of $A(F)$.*

Using the given basis for F and any basis for G, we make explicit identifications $A(F/F') \leftrightarrow GL(n, \mathbb{Z})$ and $A(G/G') \leftrightarrow GL(n, \mathbb{Z})$, and thus obtain exact sequences

$$1 \longrightarrow K(F) \longrightarrow A(F) \xrightarrow{\pi_F} \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1$$

$$1 \longrightarrow K(G) \longrightarrow A(G) \xrightarrow{\pi_G} \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1.$$

Since G is a normal subgroup of $A(F)$, $A(F)$ acts on G by conjugation. By convention, $A(G)$ acts on G by conjugation, since we identify G with the inner automorphisms in $A(G)$. Thus there is an induced homomorphism $\alpha: A(F) \to A(G)$.

LEMMA 3. (a) *$\alpha: A(F) \to A(G)$ is one-to-one.*

(b) $\alpha(K(F)) \subseteq K(G)$.

Proof. (a) $\operatorname{Ker} \alpha = C$, the centralizer of G in $A(F)$. Note that the centralizer of F in $A(F)$ is trivial, by the definition of $A(F)$.

We claim that $C = 1$. For if $C \neq 1$, then $C \cap F \supseteq [C, F] \neq 1$ and $G \cap F \supseteq [G, F] \neq 1$. Thus $C \cap F$ and $G \cap F$ are nontrivial normal subgroups of F which centralize each other, which is impossible since F is a non-abelian free group.

(b) By [D-F, Theorem B], any homomorphism $A(F) \to \mathrm{GL}(n, \mathbb{Z})$ contains $K(F)$ in its kernel. Apply it to the composition

$$A(F) \xrightarrow{\alpha} A(G) \longrightarrow A(G)/K(G) \cong \mathrm{GL}(n, \mathbb{Z}). \quad \blacksquare$$

In the sequel, we will suppress α and regard $A(F)$ as a subgroup of $A(G)$, and $K(F)$ as a subgroup of $K(G)$. Via the fixed identifications $A(F)/K(F) \cong A(F/F') \leftrightarrow \mathrm{GL}(n, \mathbb{Z})$ and $A(G)/K(G) \cong A(G/G') \leftrightarrow \mathrm{GL}(n, \mathbb{Z})$ we obtain an induced homomorphism $\bar{\alpha}\colon \mathrm{GL}(n, \mathbb{Z}) \to \mathrm{GL}(n, \mathbb{Z})$ which makes the following diagram commute:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & K(F) & \longrightarrow & A(F) & \xrightarrow{\pi_F} & \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \bar{\alpha}} & & \\
1 & \longrightarrow & K(G) & \longrightarrow & A(G) & \xrightarrow{\pi_G} & \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & 1
\end{array}$$

The unlabeled maps on the left side of the diagram are inclusion maps.

LEMMA 4. (a) Ker $\bar{\alpha}$ *contains no elements of order* 2.

(b) $\bar{\alpha}(-I) = -I$, *where I is the $n \times n$ identity matrix.*

Proof. (a) Suppose that $\phi \in \operatorname{Ker} \bar{\alpha}$, $\phi^2 = 1$. By [H-R, Lemma 1] every element of $\mathrm{GL}(n, \mathbb{Z})$ of order $\leqslant 2$ is conjugate to a diagonal sum of blocks

$$1, \qquad -1, \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Such diagonal sums are easily seen to be the images of elements of order $\leqslant 2$ in $A(F)$, so any element of $\mathrm{GL}(n, \mathbb{Z})$ of order $\leqslant 2$ can be lifted to an element of order $\leqslant 2$ in $A(F)$. Hence there is some $\phi_0 \in A(F)$ such that $\phi_0^2 = 1$ and $\pi_F(\phi_0) = \phi$.

Since $\pi_G(\phi_0) = \bar{\alpha}\pi_F(\phi_0) = \bar{\alpha}(\phi) = 1$, $\phi_0 \in K(G)$. But $K(G)$ is torsion-free (Theorem 1(d)) and $\phi_0^2 = 1$. Thus $\phi_0 = 1$, and $\phi = \pi_F(\phi_0) = 1$.

(b) Let $D \cong \mathbb{Z}_2^n$ be the subgroup of diagonal matrices in $\mathrm{GL}(n, \mathbb{Z})$. By (a), Ker $\bar{\alpha}$ contains no elements of order 2, so $\bar{\alpha}\colon D \to \mathrm{GL}(n, \mathbb{Z})$ is one-to-one. By the representation theory of finite groups, there is a matrix $P \in \mathrm{GL}(n, \mathbb{Q})$ such that $P\bar{\alpha}(D)\, P^{-1} = D$. This implies that there is a *unique* $d \in D$ such that $\bar{\alpha}(d) = -I$. Every nontrivial $d \in D$ except $-I$ is conjugate in $\mathrm{GL}(n, \mathbb{Z})$ to some $d' \in D$, $d \neq d'$. Hence the uniqueness of d forces $d = -I$. $\blacksquare$

LEMMA 5. *Suppose that $\phi \in A(G)$, $\phi^2 = 1$, and $\pi_G(\phi) = -I$. Then there exists a free basis $g_1, ..., g_n$ for G such that $\phi(g_i) = g_i^{-1}$.*

Proof. Dyer and Scott [D-S, Theorem 3] have classified automorphisms of prime order of a free group of finite rank. Their result for the prime 2 says that G is a free product of ϕ-stable free groups G_t upon which ϕ acts in one of the following ways:

(i) $G_t = \langle g \rangle$, $\phi(g) = g^{-1}$;

(ii) $G_t = \langle g \rangle$, $\phi(g) = g$;

(iii) $G_t = \langle g, h \rangle$, $\phi(g) = h$, $\phi(h) = g$;

(iv) $G_t = \langle g, h_1, ..., h_s, s \geqslant 1 \rangle$, $\phi(g) = g^{-1}$, $\phi(h_i) = gh_ig^{-1}$.

The hypothesis $\pi_G(\phi) = -I$ permits only the first action (i) to occur, and this gives the desired basis. ∎

LEMMA 6. $G \subseteq K(F)$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} G & \longrightarrow & A(F) & \xrightarrow{\pi_F} & \mathrm{GL}(n, \mathbb{Z}) \\ & & \big\downarrow & & \big\downarrow{\scriptstyle \bar{\alpha}} \\ & & A(G) & \xrightarrow{\pi_G} & \mathrm{GL}(n, \mathbb{Z}) \end{array}.$$

Let θ be the automorphism of F defined by $\theta(x_i) = x_i^{-1}$. Then $\pi_F(\theta) = -I$, so by Lemma 4(b)

$$\pi_G(\theta) = \bar{\alpha}\pi_F(\theta) = \bar{\alpha}(-I) = -I.$$

By Lemma 5, there is a free basis $g_1, ..., g_n$ for C such that $\theta g_i \theta^{-1} = \theta(g_i) = g_i^{-1}$ for $i = 1, ..., n$.

Now $\theta^2 = (\theta g_i)^2 = 1$. Since $\pi_F(\theta) = -I$ which is central in $\mathrm{GL}(n, \mathbb{Z})$, $\pi_F(g_i)^2 = 1$. But $\bar{\alpha}\pi_F(g_i) = \pi_G(g_i) = 1$, since $\pi_G(G) = 1$. By Lemma 4(a), $\operatorname{Ker} \bar{\alpha}$ contains no elements of order 2. Thus $\pi_F(g_i) = 1$, and $G \subseteq K(F) = \operatorname{Ker} \pi_F$, since $g_1, ..., g_n$ generate G. ∎

LEMMA 7. $GK_2(F) = FK_2(F)$.

Proof. By Lemma 3(b), $K(F) \subseteq K(G)$; by Lemma 6, $G \subseteq K(F)$; by

Theorem 1(e), $K_2(F) = K(F)'$ and $K_2(G) \subseteq K(G)'$. Hence there is a commutative diagram of canonical homomorphisms,

$$\begin{tikzcd} G/G' \arrow[r, "i"] \arrow[dr, "k"'] & K(F)/K_2(F) \arrow[d, "j"] \\ & K(G)/K_2(G) \end{tikzcd};$$

$$i(G/G') = GK_2(F)/K_2(F);$$

$$k(G/G') = GK_2(G)/K_2(G).$$

By Theorem 2, $GK_2(G)/K_2(G)$ is a rank n $\mathbb{Z}$-module direct summand of $K(G)/K_2(G)$. Hence $GK_2(F)/K_2(F)$ is a rank n $\mathbb{Z}$-module direct summand of $K(F)/K_2(F)$. Since G is normal in $A(F)$, $GK_2(F)/K_2(F)$ is $A(F)$-invariant. By Theorem 2, $FK_2(F)/K_2(F)$ is the unique $A(F)$-invariant rank n $\mathbb{Z}$-module direct summand of $K(F)/K_2(F)$. Hence

$$GK_2(F)/K_2(F) = FK_2(F)/K_2(F) \qquad \text{and} \qquad GK_2(F) = FK_2(F). \quad \blacksquare$$

LEMMA 8. $[G, G] \subseteq [F, F]$.

Proof. By Lemma 7, $G \subseteq FK_2(F)$; by Theorem 1(e), $K_2(F) = K(F)'$ and $K_2(G) = K(G)'$; by Lemma 3(b), $K(F) \subseteq K(G)$. Hence

$$[G, G] \subseteq [FK_2(F), G] \subseteq F[K_2(F), G] \subseteq F[K_2(G), G] = F\gamma_3(G).$$

We now prove by induction on i that $[G, G] \subseteq F\gamma_i(G)$ for all i. For the inductive step, assume that $[G, G] \subseteq F\gamma_i(G)$. Then

$$F\gamma_{i+1}(G) = F[F\gamma_i(G), G] \supseteq F[[G, G], G] = F\gamma_3(G) \supseteq [G, G].$$

Since $G \subseteq K(F)$ (Lemma 6), $\gamma_i(G) \subseteq \gamma_i(K(F)) \subseteq K_i(F)$ (Theorem 1(c)). Then Theorem 1(f) yields

$$[G, G] \subseteq \bigcap F\gamma_i(G) \subseteq \bigcap FK_i(F) = F.$$

Finally, since $[G, G] \subseteq K_2(F)$, Theorem 1(b) implies that

$$[G, G] \subseteq K_2(F) \cap F = [F, F]. \quad \blacksquare$$

THEOREM 9. *Let F be a free group of finite rank $n \geqslant 2$, let $A(F)$ be its automorphism group, and identify F with the subgroup of $A(F)$ of inner automorphisms. Suppose that G is a normal subgroup of $A(F)$ and that G is free of rank n. Then $G = F$.*

Proof. By Theorem 1(g),

$$F = \{\phi \in K(F) \mid [\phi, K(F)] \subseteq F'\},$$
$$G = \{\phi \in K(G) \mid [\phi, K(G)] \subseteq G'\}.$$

By Lemma 3(b), $K(F) \subseteq K(G)$, and by Lemma 8, $G' \subseteq F'$. Thus

$$g \in G \Rightarrow [g, K(G)] \subseteq G' \Rightarrow [g, K(F)] \subseteq F' \Rightarrow g \in F,$$

so $G \subseteq F$. Then G is a finitely generated normal subgroup of F, a free group of finite rank, so G has finite index in F [M-K-S, Theorem 2.10, p.104]. The Schreier formula

$$\operatorname{rank} G = [G : F](\operatorname{rank} F - 1) + 1$$

shows that $[G : F] = 1$, so $G = F$. ▮

References

[A] S. Andreadakis, On the automorphisms of free groups and free nilpotent groups, *Proc. London Math. Soc.* **15** (1965), 239–268.

[B] S. Bachmuth, Induced automorphisms of free groups and free metabelian groups, *Trans. Amer. Math. Soc.* **122** (1966), 1–17.

[Bu] W. Burnside, "Theory of Groups of Finite Order," Dover, New York, 1955.

[D-F] J. L. Dyer and E. Formanek, The automorphism group of a free group is complete, *J. London Math. Soc.* **11** (1975), 181–190.

[D-S] J. L. Dyer and P. G. Scott, Periodic automorphisms of free groups, *Commun. Alg.* **3** (1975), 195–201.

[G] J. A. Green, "Polynomial Representations of GL_n," Lecture Notes in Mathematics, Vol. 830, Springer-Verlag, New York/Berlin, 1980.

[H-R] L. K. Hua and I. Reiner, Automorphisms of the unimodular group, *Trans. Amer. Math. Soc.* **71** (1951), 331–348.

[M-K-S] W. Magnus, A. Karrass, and D. Solitar, "Combinatorial Group Theory," Wiley Interscience, New York, 1966.